JOURNAL OF PROPULSION AND POWER
Vol. 11, No. 2, March–April 1995

Thrust Formula for Applied-Field Magnetoplasmadynamic Thrusters Derived from Energy Conservation Equation

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A thrust formula for applied-field magnetoplasmadynamic (MPD) thrusters has been derived from the energy conservation equation for the propellant plasma flow by assuming that the work done by electromagnetic forces is converted into kinetic energy of the exhaust plasma. Three acceleration mechanisms, 1) generalized Hall acceleration, 2) swirl acceleration, and 3) self-magnetic acceleration are taken into account. This formula enables theoretical calculation of thrust of an applied-field MPD thruster from controllable thruster operation parameters, and estimation of the contributions of the respective acceleration mechanisms.

Nomenclature

a	= constant
$a_{\bar{r}}$	= $\bar{r}/r_a$
B	= magnetic field
b	= constant
c	= constant
E	= electric field
E'	= effective electric field
E'_{hall}	= effective Hall electric field
e	= charge of electron
J	= total discharge current
j	= current density
l	= length of acceleration region
M_A	= ratio of exhaust velocity to Alfvén velocity
m_i	= ion mass
$\dot{m}$	= mass flow rate
n_e	= electron number density
p_e	= electron pressure
R_m	= magnetic Reynolds number
r, θ, z	= cylindrical coordinate
$\bar{r}$	= radial position of representative point
S	= coefficient related to torque, Eq. (32)
T	= thrust
u	= fluid velocity
u_{ex}	= exhaust velocity
V	= volume
V_0	= $\pi r_a^2 l$
$\{\bar{\ }\}$	= representative value
α	= constant
β	= $1/(n_e e)$
μ_0	= permeability in vacuum
ξ	= ψ/ψ_c
ρ	= density
σ	= electrical conductivity
ϕ	= B_r/B_z
ψ	= $\bar{\sigma}B^2/\dot{m}$
$\omega_e\tau_e$	= electron Hall parameter

Subscripts

a	= anode
c	= cathode (r_c), critical (B_c, ψ_c)
e	= electron
ef	= effective
ex	= exhaust
hall	= Hall effect, generalized Hall acceleration
self	= self-magnetic acceleration
swirl	= swirl acceleration

I. Introduction

IN order to realize large-sized, long-term space missions such as a Mars mission, a high-powered, high-I_{sp} propulsion device needs to be developed. For the above requirements an MPD thruster is thought to be suitable, owing to its high-power potentiality, high-thrust density, structural simplicity, and high-I_{sp} (higher than a couple of 1000 s) operation ability. Steady-state operation is thought favorable to achieve diffuse current attachment and low erosion rates. However, in the input power range of tens of kilowatts, the discharge current is not large enough to obtain a large thrust component due to self-induced magnetic field, which is in proportion to the square of the discharge current.[1] In order to increase the performance of an MPD thruster in the above operation range, an external magnetic field needs to be applied in the acceleration region.

Sasoh and Arakawa[2] found that when the thruster operates under a high applied-field strength and low mass flow rate condition, the effect of azimuthal induced current on thrust production becomes dominant. They obtained the thrust formula for this acceleration mechanism and found that the specific impulse is characterized by a characteristic parameter $B^2/\dot{m}$.

The interaction between the applied magnetic field and the discharge current results in azimuthal electromagnetic force, which causes the swirl motion of the propellant plasma. The swirl kinetic energy can be converted into the axial kinetic energy both through the solid nozzle of the thruster (the centrifugal force is to be balanced with the static pressure gradient), and the magnetic nozzle formed by the applied magnetic field. Fradkin et al.[3] give the thrust formula for this acceleration mechanism. The formula is based on the conservation of the angular momentum and that of the kinetic energy of the propellant plasma flow.

When the discharge current becomes large enough, the interaction force between the self-induced magnetic field and

Received Oct. 11, 1991; revision received Nov. 30, 1993; accepted for publication June 6, 1994.

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the discharge current cannot be neglected. The thrust formula is algebraically obtained by Jahn.[1] The formula is obtained by integrating the electromagnetic interaction force over the interelectrode region. An MPD thruster can obtain high thrust performance by increasing the discharge current through quasisteady operation (self-field MPD thruster), etc. Moreover, Tahara et al.[4] experimentally found that an applied magnetic field at an appropriate strength enhances the thrust performance of the self-field MPD thruster. This enhancement was thought not due to increasing the self-field thrust component, but due to other components related directly to the applied magnetic field.

From the above descriptions, there may seem to exist three different acceleration mechanisms that are related to electromagnetic forces. In order to design the performance of an applied-field MPD thruster, however, one needs to estimate the thrust in a wide operation range of such controllable operation parameters as applied-field strength, mass flow rate, discharge current, and thruster dimensions. The purpose of this study is to obtain a general formula of the thrust produced by an applied-field MPD thruster and to make clear the interrelationships among the above acceleration mechanisms.

II. Generation of Electromagnetic Thrust in an Applied-Field MPD Thruster

Cylindrical coordinates r, θ, z are used in the derivation. An applied-field MPD thruster is assumed to have an axisymmetric structure. The flowfield of the acceleration region is also assumed axisymmetric. An external magnetic field is applied in the axial and radial direction ("applied magnetic field"). The radial component of the applied field is assumed much smaller than the axial one, corresponding to a slowly diverging magnetic nozzle configuration. The axisymmetric discharge current in the interelectrode region induces azimuthal magnetic field, which will hereafter be referred to as "self-induced magnetic field."

The total discharge current for an applied-field MPD thruster is controllable. Integrating the current density over the whole electrode (anode or cathode) surface yields a controllable constant discharge current. The integration of j_r over a semi-infinite-long cylindrical control surface in the interelectrode region equals the total discharge current. j_r is referred to as "discharge current" in this article. When a magnetic field is applied in the axial direction, azimuthal current is induced by the interaction between the applied magnetic field and the discharge current (discussed in detail later). j_θ is referred to as "induced current" here. The azimuthal current is not induced in the absence of the applied magnetic field.

In an applied-field MPD thruster, an electromagnetic thrust is generated through the interactions between a discharge or induced current component and an applied or self-induced magnetic field component. The following three components are dominant in the interaction forces:

1) Generalized Hall Acceleration: An axial electromagnetic force is produced by the interaction between the azimuthal-induced current and the radial applied magnetic field ($j_\theta B_r$). Because the azimuthal current is induced not only due to the Hall effect, but also due to the diamagnetic effect,[2] this electromagnetic acceleration will hereafter be referred to as "generalized Hall acceleration."

2) Swirl Acceleration: An azimuthal electromagnetic force is produced by the radial discharge current and the axial applied magnetic field ($j_r B_z$). This azimuthal force causes the swirl motion of the propellant plasma flow, resulting in the input of swirl kinetic energy. Although some of the swirl kinetic energy is once converted into the static enthalpy of the plasma flow, it can finally be converted into an axial kinetic energy.

3) Self-Magnetic Acceleration: The interaction between the radial discharge current and the azimuthal self-induced magnetic field produces an axial electromagnetic force ($j_r B_\theta$). The strength of the self-induced magnetic field increases proportionally to the discharge current. Therefore, this electromagnetic force scales with the square of the discharge current. When the discharge current is at low levels, this component is usually negligible.[1]

The reaction forces of electromagnetic forces on the propellant plasma flow are exerted both on the magnetic devices (electromagnets or permanent magnets or both), which produce the applied magnetic field, and on the discharge current circuit. The hydrodynamic pressure exerted on the solid surfaces of the thruster contributes a thrust component; both the static enthalpy and the swirl kinetic energy of the propellant plasma flow can be converted into an axial kinetic energy through the solid nozzle.

In order to differentiate the Hall acceleration from the swirl acceleration, the following point is important: On a z-r plane, if the plasma flows completely along applied magnetic field lines of force, a Lorentz force $j_\theta \times B$ does not do any work. In this case, according to the present definition, the generalized Hall acceleration does not contribute to thrust production at all. Since u_r is neglected here, the $j_\theta B_z$ component balances with the sum of a centrifugal force of the rotating plasma and a pressure gradient. This implies that the magnetic field lines of force act as a magnetic nozzle. In expansion processes through the magnetic nozzle, no external energy is input. Only energies input by the respective electromagnetic forces are converted into the axial kinetic energy of the exhaust plasma (thermal energy due to Joule heating is neglected here). Therefore, the contributions of the respective thrust production mechanisms are completely differentiated from each other based on the works done by the respective Lorentz force components [see Eq. (4)].

III. Derivation of Thrust Formula

The energy conversion mechanisms are, as described above, so complicated that a single thrust component cannot be isolated from the total thrust. Instead of calculating the respective thrust components separately, the total thrust is obtained from an energy conservation relation.

For simplicity, the work done by electromagnetic forces is assumed to be converted completely into the axial kinetic energy of the propellant plasma flow both through the magnetic nozzle and the solid nozzle. Neglecting the static enthalpy and viscosity of the exhaust plasma, the energy conservation equation is

$$\int (j \times B)\cdot u \,\mathrm{d}V = \frac{1}{2}\dot{m}u_{\mathrm{ex}}^2 \tag{1}$$

The current density is related to electric field, magnetic field, fluid velocity, etc., by the generalized Ohm's law

$$j = \sigma[E + \beta\nabla p_e + u \times B - \beta j \times B] \tag{2}$$

For simplicity, one assumes that $j_z/j_\theta \ll 1$. By the order estimation of the axial component of Eq. (2), j_z/j_θ is found of the order of $\sigma\beta B_r$, which, therefore, is assumed much smaller than unity here. In Eq. (2), $j_\theta B_r$ is assumed to be much larger than $j_r B_\theta$, corresponding to a large generalized Hall acceleration effect over the self-magnetic one. The azimuthal component of the current density is related to the axial one by (see Appendix A):

$$j_\theta = \sigma(\beta B_z j_r + u_z B_r + \sigma\beta B_r^2 u_\theta) \tag{3}$$

The electromagnetic force is composed of three components: 1) $j_\theta B_r$ (generalized Hall acceleration), 2) $j_r B_z$ (swirl acceleration), and 3) $j_r B_\theta$ (self-magnetic acceleration). The left side of Eq. (1) can be expressed by

$$\int (j \times B)\cdot u \,\mathrm{d}V = \int (-j_\theta B_r u_z)\,\mathrm{d}V + \int (-j_r B_z u_\theta)\,\mathrm{d}V + \int j_r B_\theta u_z \,\mathrm{d}V \tag{4}$$

In the above equation, "−" signs are needed for the works done by the respective electromagnetic forces to have positive values. u_r is neglected here.

The first term of Eq. (4) is expressed such as[2]

$$\int (-j_\theta B_r u_z)\,\mathrm{d}V = -\bar{j}_\theta \bar{B}_r \bar{u}_z a_{\mathrm{hall}} V_0 \tag{5}$$

$$V_0 \equiv \pi r_a^2 l \tag{6}$$

$$T = \dot{m}u_{\mathrm{ex}} = \frac{c_2\psi - c_3\psi^2 + c_4 + [(c_2\psi - c_3\psi^2 + c_4)^2 + (1 + c_1\psi)c_5\psi]^{1/2}}{1 + c_1\psi}\,\dot{m}J \tag{20}$$

In Eq. (5), $\bar{j}_\theta$ is a representative value of j_θ. Here, a peak value at $r = \bar{r}$ is used. The value of a_{hall}, which is the ratio of the equivalent volume for the generalized Hall acceleration to V_0, depends on j_θ distribution in the acceleration region, and is usually of the order of 0.1 to 1. A first-order estimation of a_{hall} is given in Appendix B.

Neglecting the radial and azimuthal velocity components of the exhaust plasma, the representative value of the axial fluid velocity in Eq. (5) is assumed to be

$$\bar{u}_z = \alpha_1 u_{\mathrm{ex}} \tag{7}$$

The value of α_1 will be estimated later.

The plasma is assumed to rotate as a solid object. That is, the angular velocity is assumed not to vary with a radial position.[3] The representative value of the azimuthal fluid velocity is given such that

$$\bar{u}_\theta(r) = \frac{\alpha_2 J B_z r}{\dot{m}}\frac{1 - (r_c/r_a)^2}{1 + (r_c/r_a)^2} \equiv \frac{\alpha_2' J B_z r}{\dot{m}} \tag{8}$$

The second term of the right side of Eq. (4) is calculated

$$\int (-j_r B_z u_\theta)\,\mathrm{d}V = \int_{r_c}^{r_a} [-j_r B_z \bar{u}_\theta(r)]\cdot 2\pi r l \,\mathrm{d}r = \frac{\alpha_2'' J^2 B_z^2 r_a^2}{\dot{m}} \tag{9}$$

$$\alpha_2'' = \frac{[1 - (r_c/r_a)^2]^2}{2[1 + (r_c/r_a)^2]}\alpha_2 \tag{10}$$

The last term of the right side of Eq. (4) is transformed into the product of the self-magnetic thrust[1] and the representative value of the exhaust velocity

$$\int j_r B_\theta u_z \,\mathrm{d}V = b\frac{\mu_0 J^2}{4\pi}\alpha_1 u_{\mathrm{ex}} \tag{11}$$

where

$$b = \ln(r_a/r_c) + b' \tag{12}$$

In Eq. (12), b' varies from $\frac{1}{4}$ to $\frac{3}{4}$, depending on the current distribution on the cathode.

Here, one changes the notations B_z and $\bar{B}_r$ into B and $\bar{\phi}B$, respectively. Substituting Eqs. (5), (9), and (11) for Eq. (1) yields the following quadratic equation:

$$(1 + c_1\psi)u_{\mathrm{ex}}^2 - 2(c_2\psi - c_3\psi^2 + c_4)Ju_{\mathrm{ex}} - c_5\psi J^2 = 0 \tag{13}$$

Here

$$\psi = \bar{\sigma}B^2/\dot{m} \tag{14}$$

$$c_1 = 2\alpha_1^2\bar{\phi}^2 a_{\mathrm{hall}} V_0 \tag{15}$$

$$c_2 = \alpha_1 \bar{\beta}\bar{\phi} a_{\mathrm{hall}} V_0/2\pi\bar{r}l \tag{16}$$

$$c_3 = \alpha_1\alpha_2'\bar{\beta}\bar{\phi}^3\bar{r}a_{\mathrm{hall}}V_0 \tag{17}$$

$$c_4 = \alpha_1 b\mu_0 J/4\pi\dot{m} \tag{18}$$

$$c_5 = 2\alpha_2'' r_a^2/\bar{\sigma}\dot{m} \tag{19}$$

Solving Eq. (13), the thrust is calculated by Eq. (20).

Equation (20) is transformed as follows:

$$T = [(T_{\mathrm{hall}} + T_{\mathrm{self}})/2] + \{[(T_{\mathrm{hall}} + T_{\mathrm{self}})/2]^2 + T_{\mathrm{swirl}}^2\}^{1/2} \tag{21}$$

Here

$$T_{\mathrm{hall}} \equiv \frac{\psi(1 - \psi/\psi_c')}{\psi + \psi_c} T_{\mathrm{hall,max}} \tag{22}$$

$$T_{\mathrm{swirl}} \equiv \left(\frac{\psi}{\psi + \psi_c}\right)^{1/2} T_{\mathrm{swirl,max}} \tag{23}$$

$$T_{\mathrm{self}} \equiv \frac{\psi_c}{\psi + \psi_c} T_{\mathrm{self,max}} \tag{24}$$

The parameters that appear in the above equations are given by

$$\psi_c = 1/2\alpha_1^2\bar{\phi}^2 a_{\mathrm{hall}} V_0 \tag{25}$$

$$\psi_c' = 1/2\alpha_2'\bar{\phi}^2 a_{\bar{r}}^2 V_0, \quad a_{\bar{r}} \equiv \bar{r}/r_a \tag{26}$$

$$T_{\mathrm{hall,max}} = \frac{\bar{\beta}(-\bar{j}_r)\dot{m}}{\alpha_1\bar{\phi}} = \frac{\alpha_1 a_{\mathrm{hall}}}{a_{\bar{r}}}(\omega_e\tau_e)_{B=B_c} J\bar{\phi}B_c r_a \tag{27}$$

$$T_{\mathrm{swirl,max}} = (2\alpha_2'')^{1/2} J B_c r_a \tag{28}$$

$$B_c = (\dot{m}\psi_c/\bar{\sigma})^{1/2} \tag{29}$$

$$T_{\mathrm{self,max}} = 2\alpha_1 b(\mu_0 J^2/4\pi) \tag{30}$$

Equation (21) along with Eqs. (22–24) gives the general form of the thrust formula for an applied-field MPD thruster.

IV. Estimations of the Thrust Components

General forms of the momentum and energy conservation relations are

$$T = \dot{m}u_{\mathrm{ex}} \tag{31}$$

$$T\bar{u}_z + S\bar{u}_\theta = \tfrac{1}{2}\dot{m}u_{\mathrm{ex}}^2 \tag{32}$$

where $S\bar{u}_\theta$ represents a work done by a torque exerted on the plasma flow. Both T and $S\bar{u}_\theta$ have positive values. From Eqs. (31) and (32)

$$\bar{u}_z = (u_{\mathrm{ex}}/2) - (S/T)\bar{u}_\theta \tag{33}$$

Table 1 Operation conditions and calculated thrust components of some applied field MPD thrusters

Thruster, Ref.	B, T	$\dot{m}$, kg/s	J, A	ψ, m^{-3} (ψ/ψ_c)	T_{hall}, N	T_{swirl}, N	T_{self}, N	T(theory), N	T(measured), N
2, 6	0.10	9×10^{-7}	200	2.4×10^{8} (0.69)	4.4×10^{-2}	1.4×10^{-2}	1×10^{-4}	4.8×10^{-2}	5.0×10^{-2}
3	0.19	2.5×10^{-5}	350	9.7×10^{6} (0.31)	1.8×10^{-1}	6.9×10^{-1}	9×10^{-3}	7.9×10^{-1}	8.0×10^{-1}
7[a]	0.07	3.5×10^{-5}	800	8.8×10^{5} (0.0065)	3.8×10^{-2}	3.4×10^{-1}	6.4×10^{-2}	3.9×10^{-1}	5.9×10^{-1}
4[b]	0.075	2.75×10^{-3}	15,000	1.4×10^{3} (0.0005)	4×10^{0}	2.2×10^{1}	2.2×10^{1}	3.9×10^{1}	8.9×10^{1}

[a] r_a is assumed to be twice as large as the throat radius. T_e is assumed to be 2 eV.
[b] The coil configuration is C1-(3)-type; 5% of the total input power is assumed to be input to the electron translational energy.

The second term of Eq. (33) has a negative value or becomes zero. Comparing Eqs. (7) and (33)

$$\alpha_1 \begin{cases} = \frac{1}{2}, & \bar{u}_\theta = 0 \\ < \frac{1}{2}, & \bar{u}_\theta > 0 \end{cases} \tag{34}$$

Without the swirl motion of the plasma flow, α_1 theoretically equals $\frac{1}{2}$. Under modest conditions, $\alpha_1 = \frac{1}{2}$ is still a reasonable approximation.

When $\psi << \psi_c$, i.e., at a weak applied magnetic field, Eq. (23) becomes

$$T_{swirl} = (2\alpha_2'')^{1/2} J B r_a \tag{35}$$

In order for Eq. (35) to be equal to the Fradkin's formula,[3] it is needed that

$$\alpha_2 = \frac{1}{2} \tag{36}$$

From the radial component of Eq. (2)

$$\bar{j}_r \simeq \bar{\sigma}\left(\bar{E}_r + \bar{\beta}\frac{d\bar{p}_e}{dr}\right) \equiv \bar{\sigma}\bar{E}_r' \tag{37}$$

Hence, the exhaust velocity calculated from Eq. (27) is approximately given by

$$u_{hall,max} \equiv T_{hall,max}/\dot{m} \simeq 2|\bar{E}'_{hall,\theta}|/\bar{B}_r \tag{38}$$

$$\bar{E}'_{hall,\theta} = \overline{\omega_e \tau_e}\bar{E}_r' \tag{39}$$

Equation (38) implies that the exhaust velocity only due to the generalized Hall acceleration is twice as high as $\bar{E}'_{hall,\theta} \times \bar{B}_r$ drift velocity.

The dependence of $\bar{E}_r'$ on B is determined both by the impedance of an MPD thruster and by the characteristics of the power supply for the arc discharge. Generally, an applied-field MPD thruster is connected with a power supply with a constant-current profile. In this case, $\bar{E}_r'$ varies with B. However, e.g., if the power supply has a constant-voltage profile, the thruster operates as an electrostatic acceleration thruster, being identical with a "Hall-current accelerator" or a "closed-drift thruster."[5]

The exhaust velocity is an important parameter to the generalized Hall acceleration. As seen from Eqs. (22) and (27), it is expressed just by the characteristic parameter $\psi = \bar{\sigma}B^2/\dot{m}$. The ratio ψ/ψ_c can be a criterion whether the effect of the generalized Hall acceleration is large or not. Under a typical condition that $\psi_c' = 2\psi_c$ (from the measurement, $r_c/r_a = 0.38$, $a_{hall} = 0.58$, and $a_{\bar{r}} = 0.44$), T_{hall} becomes maximum at $\psi = 0.73\psi_c$.

V. Operation Regimes

Table 1 shows the typical experimental operation parameters and the respective thrust components that are calculated by this theory in the cases of some applied-field MPD thrusters.

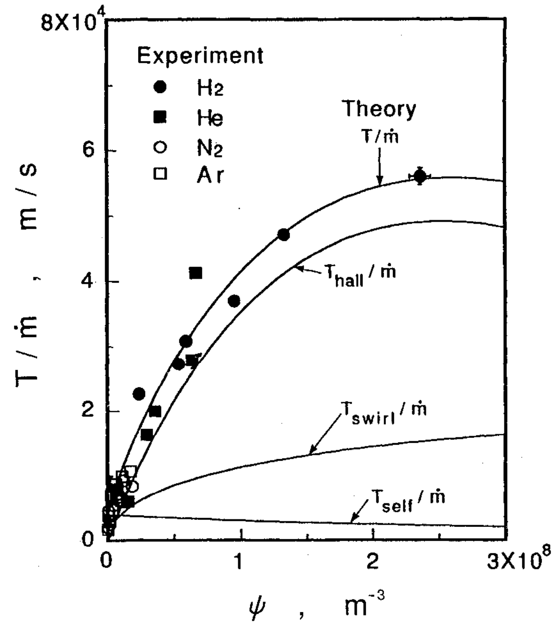

Fig. 1 $T/\dot{m} - \bar{\sigma}B^2/\dot{m}$ relation for the applied-field MPD thruster of Refs. 2 and 6. The theoretical lines are drawn under the following condition: $\bar{r} = 3.5 \times 10^{-3}$ m, $r_{a,ef} = 8 \times 10^{-3}$ m, $r_{c,ef} = 3 \times 10^{-3}$ m, $\bar{\phi} = 5 \times 10^{-2}$, $\bar{\sigma}\dot{m} = 2 \times 10^{-2}$ kg/(Ω ms), $\bar{\beta} = 5 \times 10^{-2}$ m^3/C, $l = 2 \times 10^{-2}$ m, $\bar{j}_r = c_c J/(2\pi\bar{r}l)$ A/m^2, $c_c = 0.2$, $J = 200$ A.

In the thruster operation regime performed by Sasoh and Arakawa (University of Tokyo),[2,6] it is seen that the effect of the generalized Hall acceleration is dominant. In this case, $T/\dot{m}$, which is equivalent to the average exhaust velocity, is expressed by the characteristic parameter ψ (see Fig. 1). In order to obtain the exhaust velocity as the general function of the characteristic parameter that is applicable for different kinds of propellant species, $B^2/\dot{m}$—the characteristic parameter in Ref. 2—needs to be multiplied by the electrical conductivity. In the characteristic parameter ψ, to what extent the azimuthal current is induced is estimated by $\bar{\sigma}B$, which scales with the electron Hall parameter. The thruster operation is made at relatively high-B, low-$\dot{m}$ levels (compare the ψ with the others). The ratio ψ/ψ_c is 0.69. The effect of the swirl acceleration is not large here, about 10% thrust increase. The effect of the self-magnetic acceleration is almost negligible.

In the operation regime performed by Fradkin et al.[3] (Los Alamos Scientific Laboratory), the effect of the swirl acceleration is found dominant. The maximum value of ψ set in the experiment is only 4% of that in the first case. In other words, the applied magnetic field at the input propellant mass flow rate is weak in comparison with the former case.

Under the condition that $\psi = \xi\psi_c$ and $\psi_c' = 2\psi_c$, the ratio of T_{hall} to T_{swirl} is calculated from Eqs. (22) and (23) as follows:

$$\frac{T_{\text{hall}}}{T_{\text{swirl}}} = c_6(\xi)\left(\frac{\bar{\sigma}\bar{\rho}u_{\text{ex}}\bar{\beta}^2}{l}\right)^{1/2} = c_6(\xi)\,\frac{\overline{\omega_c\tau_e}M_A}{R_m^{1/2}} \tag{40}$$

$$c_6(\xi) = \left(\frac{\xi}{\xi+1}\right)^{1/2}\left(1-\frac{\xi}{2}\right)\frac{1}{a_F}\left(\frac{a_{\text{hall}}}{8\alpha_2''}\right)^{1/2} \tag{41}$$

$$M_A \equiv \frac{u_{\text{ex}}}{B/(\mu_0\bar{\rho})^{1/2}} \tag{42}$$

$$R_m \equiv \bar{\sigma}\mu_0 u_{\text{ex}} l \tag{43}$$

At $\xi = 0.5$, $c_6(\xi)$ is maximum and is of the order of 0.5. As seen from Eq. (40), the ratio increases with increasing exhaust velocity. Moreover, when the propellant mass flow rate is decreased, both $\bar{\sigma}$ and $\bar{\rho}\bar{\beta}^2$ [$=m_i/(e^2n_e)$ for a fully ionized plasma] will increase, thereby enhancing the effect of the generalized Hall acceleration. Also, the axial length of the acceleration region is to be small so that the contribution of the generalized Hall acceleration is large.

Also under the operation condition done by Connolly et al.[7] (NASA Lewis Research Center), the swirl acceleration is dominant. In this case, ψ/ψ_c is much smaller than unity. T_{hall} is one order of magnitude smaller than T_{swirl}.

Tahara et al.[4] (Osaka University) combine a self-field MPD thruster with an applied magnetic field coil connected in series with the discharge circuit. In this case, the applied magnetic field is the function of the discharge current. Since the external magnetic field is applied so that it becomes as strong as the self-induced magnetic field in the main region for current conduction, the T_{self} and T_{swirl} are almost equal. From Eqs. (23) and (24), the ratio of T_{self} to T_{swirl} under the condition that $\psi \ll \psi_c$ is calculated:

$$T_{\text{self}}/T_{\text{swirl}} = c_7(B_{\theta,r_a}/B)(\psi \ll \psi_c) \tag{44}$$

$$B_{\theta,r_a} = \mu_0 J/2\pi r_a \tag{45}$$

$$c_7 = b/2^{3/2}\alpha_2''^{1/2} \tag{46}$$

The value of c_7 is of the order of unity. Therefore, it is found from Eq. (44) that the ratio of these components approximately equals the ratio of the self-induced to applied magnetic fields.

In the last two cases, the effect of the generalized Hall acceleration is negligible. The values of the characteristic parameter are two and five orders of magnitude smaller than that of the first case, respectively. In these cases, the measured thrusts are much larger than the calculated ones. This is thought to be due to an electrothermal effect that is not taken into account in this study.

VI. Conclusions

The formula that gives the thrust produced by an applied-field MPD thruster from the controllable thruster operation parameters has theoretically been obtained; the formula is derived from the energy conservation equation of the propellant plasma flow. This formula is applicable to thruster operation conditions under which the effect of electrothermal acceleration is negligible. Using this formula, the relative contributions of the generalized Hall acceleration, swirl acceleration, and self-magnetic acceleration can be estimated. The relative contributions of the self-magnetic and swirl accelerations are found to be estimated by the ratio of the self-induced to applied magnetic field strengths. At a moderate applied-field strength, i.e., when ψ is comparable with ψ_c, a large effect of the generalized Hall acceleration is obtained under the conditions that the exhause velocity is high, the propellant mass flow rate is low and the axial length of the acceleration region is small, thereby suggesting that the generalized Hall acceleration is suitable for small-sized, high-I_{sp} thruster operations.

Appendix A: Derivation of Eq. (3)

Here, j, B, u, and E are assumed to have the following components, respectively:

$$j = \begin{bmatrix} j_r \\ j_\theta \\ 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_r \\ B_\theta \\ B_z \end{bmatrix}, \quad u = \begin{bmatrix} 0 \\ u_\theta \\ u_z \end{bmatrix}$$
$$E = \begin{bmatrix} E_r \\ 0 \\ 0 \end{bmatrix}, \quad \nabla p_e = \begin{bmatrix} \frac{\mathrm{d}p_e}{\mathrm{d}r} \\ 0 \\ 0 \end{bmatrix} \tag{A1}$$

From Eqs. (2) and (A1)

$$j_z = \sigma[-u_\theta B_r + \beta(j_\theta B_r - j_r B_\theta)] \tag{A2}$$

$$j_\theta = \sigma[u_z B_r - \beta(j_z B_r - j_r B_z)] \tag{A3}$$

Assuming that $j_r B_\theta \ll j_\theta B_r$, Eqs. (A2) and (A3) are combined and transformed into

$$j_\theta = [\sigma/1 + (\sigma\beta B_r)^2](\beta B_z j_r + u_z B_r + \sigma\beta B_r^2 u_\theta) \tag{A4}$$

Under the condition that $\sigma\beta B_r \ll 1$

$$j_\theta = \sigma(\beta B_z j_r + u_z B_r + \sigma\beta B_r^2 u_\theta) \tag{A5}$$

Appendix B: First-Order Estimation of a_{hall}

Substituting $\bar{u}_z$ for u_z on the left side of Eq. (5), for simplicity, yields

$$\int (-j_\theta B_r)\,\mathrm{d}V = -\bar{j}_\theta \bar{B}_r a_{\text{hall}} V_0 \tag{B1}$$

From the measurements of j_θ distribution conducted using a Hall-effect current sensor,[2] it is reasonable to assume a j_θ distribution to a first-order approximation as follows:

$$j_\theta = \begin{cases} 2r\bar{j}_\theta/r_a, & 0 \le r \le r_a/2 \\ 2(r_a - r)\bar{j}_\theta/r_a, & r_a/2 \le r \le r_a \end{cases} \tag{B2}$$

This approximation corresponds to a typical value of a_F, $\frac{1}{2}$. B_r is given by

$$B_r = 2\bar{\phi}Br/r_a \tag{B3}$$

From Eqs. (6) and (52–54)

$$a_{\text{hall}} = \frac{1}{\bar{j}_\theta\bar{\phi}B\pi r_a^2 l}\int_0^{r_a} j_\theta B_r\cdot 2\pi r l\,\mathrm{d}r = \frac{7}{12} \simeq 0.58 \tag{B4}$$

Acknowledgments

This work was supported by funding from the Ishida Foundation, Nagoya, Japan. This work was partially conducted while the first author belonged to the Department of Aeronautical Engineering, Nagoya University, Nagoya, Japan. The authors would like to thank them for offering us the opportunity of starting this study.

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